Exercise 35

(a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire in the shape of a space curve C if the wire has density function $\rho(x, y, z)$.

(b) Find the center of mass of a wire in the shape of the helix $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \leq t \leq 2\pi$, if the density is a constant k.

Solution

Calculate the x-coordinate of the center of mass.

$$\bar{x} = \frac{\displaystyle\int x\,dm}{\displaystyle\int dm} = \frac{\displaystyle\int_C x(\rho\,ds)}{\displaystyle\int_C \rho\,ds} = \frac{\displaystyle\int_0^{2\pi} x(t)k\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}{\displaystyle\int_0^{2\pi} k\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}$$

$$= \frac{k\displaystyle\int_0^{2\pi} (2\sin t)\sqrt{(2\cos t)^2 + (-2\sin t)^2 + (3)^2}\,dt}{k\displaystyle\int_0^{2\pi} \sqrt{(2\cos t)^2 + (-2\sin t)^2 + (3)^2}\,dt}$$

$$= \frac{\displaystyle\int_0^{2\pi} (2\sin t)\sqrt{13}\,dt}{\displaystyle\int_0^{2\pi} \sqrt{13}\,dt}$$

$$= \frac{2\displaystyle\int_0^{2\pi} \sin t\,dt}{\displaystyle\int_0^{2\pi} dt}$$

$$= 0$$

Calculate the y-coordinate of the center of mass.

$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_C y(\rho\,ds)}{\int_C \rho\,ds} = \frac{\int_0^{2\pi} y(t)k\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}{\int_0^{2\pi} k\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}$$

$$= \frac{k\int_0^{2\pi} (2\cos t)\sqrt{(2\cos t)^2 + (-2\sin t)^2 + (3)^2}\,dt}{k\int_0^{2\pi} \sqrt{(2\cos t)^2 + (-2\sin t)^2 + (3)^2}\,dt}$$

$$= \frac{\int_0^{2\pi} (2\cos t)\sqrt{13}\,dt}{\int_0^{2\pi} \sqrt{13}\,dt}$$

$$= \frac{2\int_0^{2\pi} \cos t\,dt}{\int_0^{2\pi} dt}$$

$$= 0$$

Calculate the z-coordinate of the center of mass.

$$\bar{z} = \frac{\int z\,dm}{\int dm} = \frac{\int_C z(\rho\,ds)}{\int_C \rho\,ds} = \frac{\int_0^{2\pi} z(t)k\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}{\int_0^{2\pi} k\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}$$

$$= \frac{k\int_0^{2\pi} (3t)\sqrt{(2\cos t)^2 + (-2\sin t)^2 + (3)^2}\,dt}{k\int_0^{2\pi} \sqrt{(2\cos t)^2 + (-2\sin t)^2 + (3)^2}\,dt}$$

$$= \frac{\int_0^{2\pi} (3t)\sqrt{13}\,dt}{\int_0^{2\pi} \sqrt{13}\,dt}$$

$$= \frac{3\int_0^{2\pi} t\,dt}{\int_0^{2\pi} dt}$$

$$= 3\pi$$

Therefore, the center of mass is $(0, 0, 3\pi)$.